ISI – Bangalore Center – B Math - Physics IV – Mid Semester Exam
Date: 26 February 2018. Duration of Exam: 3 hours
Total marks: 80

UNLESS otherwise mentioned, units are chosen are such that c=1. The Greek indices μ,ν take values 0,1,2,3 and the Latin indices *i,j* take values 1,2,3. The contravariant position four vector is denoted by $x^\mu = (x^0 = ct = t, x^1 = x, x^2 = y, x^3 = z)$ and the Lorentz transformation is represented by the matrix L^μ_ν where μ,ν represent the rows and columns respectively. All Lorentz transformations are proper. The metric $g_{\mu\nu}$ is diagonal with $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$. Also assume that the frames S and S' which are moving with respect to each other have their axes aligned and that
$x^0 = x^1 = x^2 = x^3 = 0 \Leftrightarrow x'^0 = x'^1 = x'^2 = x'^3 = 0$

Q1. [Total Marks: 4+4+4+3=15]

a.) Let Q be a spacetime point. Let P be another spacetime point which lies outside the light cone of P. Show that there exists different frames in which each of the following statements can be true.

i) $t_P > t_Q$

ii) $t_P = t_Q$

iii) $t_P < t_Q$

b.) Let a train car with rest length L move in the x direction with respect to the ground frame with speed very small compared to the speed of light. A ball is thrown in the train from one end of the compartment along the x direction towards the other end of the compartment. In the frame of the train, the ball moves with constant speed also small compared to c and hits the other end of the compartment. Let these two events of the ball being thrown at one end and hitting the other end be named P and Q.

Prove or disprove the following statement: There is a frame of reference moving with respect to the ground in which P and Q are simultaneous events.

Q2. [Total Marks:4+4+2+5=15]

A particle moves with velocity $\vec{u}$ and $\vec{u}'$ as seen from two frames S and S'. S' is moving with respect to S with constant velocity $\vec{v}$.

a.) Derive the expressions of $\vec{u}'$ in terms of $\vec{u}$ and γ_v when $\vec{v} = (v,0,0)$

b.) For an arbitrary $\vec{v}$, derive the formula $\vec{u}'_{parallel} = \dfrac{\vec{u}_{parallel} - \vec{v}t}{\sqrt{\left(1 - \dfrac{\vec{v}.\vec{u}'}{c^2}\right)}}$

c.) Write without derivation the corresponding formula for $\vec{u}'_{perp}$.

d.) In the lab frame, two particles are moving with the same speed u along two straight lines which are perpendicular to each other. Show that the speed of anyone of them as seen from the other is given by $u\sqrt{(2-u^2)}$. Hint: This problem becomes simple if you use four vectors

Q3. [Total Marks: 4+4+2+5=15]

a.) Show that if any one of the components of a four vector is zero in all frames then it must be a zero vector.

b.) Let there be n particles with momentum four vector $p_1^\mu, p_2^\mu, \ldots, p_n^\mu$ with $p_\alpha^\mu = (p_\alpha^0 = m_\alpha \gamma_{u_\alpha}, p_\alpha^i = m_\alpha \gamma_{u_\alpha} u_\alpha^i)$. Here α denotes the α-th particle with rest mass m_α with speed u_α^i. Define $P_{tot}^\mu = \sum_\alpha p_\alpha^\mu$. Using the result of part a.) above, show that if any of the components of the total four momenta is conserved in all frames connected by a Lorentz transformation, then the all components of the four momenta are conserved.

c.) Determine if the following statement is true or false: In the frame where the total three momenta is zero, its energy component is the sum of the rest masses of the particles.

d.) A particle of rest mass m moving with speed v hits an identical particle at rest. The two masses fuse and create a single particle of rest mass M. Show that $M = 2m\sqrt{\dfrac{1+\gamma}{2}}$

Q4. [Total Marks: 2+3+5=10]

The current vector in elcgtromagnetism is defined is $\vec{j} = \rho\vec{u}$ where ρ is the local charge density moving with velocity $\vec{u}$.

a.) Show that ρ is related to ρ_0, the charge density in the instantaneous rest frame, by the equation: $\rho = \gamma_u \rho_0$

b.) Show that $(c\rho, \vec{j})$ transforms as a four vector under Lorentz transformation.

c.) Show that $\frac{1}{c}\frac{\partial \rho}{\partial t} + \vec{\nabla}.\vec{j} = 0$ is a Lorentz invariant statement. [For this you need to consider and demonstrate how the differential operator in the equation transforms]

Q 5. [Total Marks: 6+9=15]

Recall that $(\gamma_u \frac{\vec{F}.\vec{u}}{c}, \gamma_u \vec{F})$ are the components of the force four vector.

a.) Show that the force $\vec{F}'$ in the instantaneous rest frame of the particle is related to the force $\vec{F}$ as measured in another frame by

$\vec{F}'_{parallel} = \vec{F}_{parallel}$ and $\vec{F}'_{perp} = \gamma \vec{F}_{perp}$

b.) Using the result in part a.) to determine how the electric field components in the rest frame are related to the Electric and Magnetic fields in another frame which is moving in the x direction.

Q 6. [Total Marks: 6+4=10]

[In the class we have seen how the x components of the electric and magnetic fields are unchanged under boost in the x direction and how this is related to the transformation properties of an antisymmetric tensor under the same transformation. The following problem follows that line of thinking]

Let $(ct, x(t), y(t), z(t))$ represent the trajectory of a particle in a particular frame and let $(\frac{E}{c}, p^x, p^y, p^z)$ be its four momenta.

Show that under a boost in the x direction

a.) The quantity $(ctp^x - \frac{Ex}{c})$ remains unchanged.

b) The quantities yp^y, yp^z, $(yp^z - zp^y)$ also remain unchanged.

Hint: Relate the above objects to a tensor $x^\mu p^\nu$ or a variation of it. For part b.) you can use an even simpler argument.

Results you may use.

Under a boost $\vec{v} = (v^1, v^2, v^3)$, a four vector $a^\mu = (a^0, \vec{a})$ transforms as

$$a'^0 = \gamma\,(a^0 - \vec{\beta}\cdot\vec{a})$$

$$\vec{a}'_{\parallel} = \gamma\,(\vec{a}_{\parallel} - \vec{\beta}\, a^0)$$

$$\vec{a}'_{\perp} = \vec{a}_{\perp}$$

where $\vec{a} = \vec{a}_{\perp} + \vec{a}_{\parallel}$ and $\vec{\beta} = \frac{\vec{v}}{c}$.